Painlevé analysis and bright solitary waves of the higher-order nonlinear Schrödinger equation containing third-order dispersion and self-steepening term

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A general form of the higher-order nonlinear Schrödinger equation that includes terms accounting for the third-order dispersion and the self-steepening effect has been investigated using the Painlevé singularity structure analysis in order to identify the underlying integrable models. This equation fails to pass the Painlevé test for the entire parameter space except for two specific choices of the parameters. As a consequence, it was found that two recently introduced higher-order nonlinear Schrödinger equations fail to pass the Painlevé integrability test. Moreover, one of those equations describes optical pulses with large frequency shifts as compared to the chosen carrier frequency that renders that equation inappropriate for describing femtosecond soliton propagation in monomode optical fibers. Another equation is introduced and bright solitary waves are provided. These solitary waves describe pulses with either very small or even zero-frequency shifts. The conditions on fiber parameters for the existence of those solitary waves are also discussed.

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I. INTRODUCTION

As first pointed out by Hasegawa and Tappert [1], the propagation of intense optical signals in nonlinear dispersive media leads to the formation of optical solitons. Optical solitons in fibers propagate, for appropriate combinations of pulse shape and intensity, without any change in their shape because the self-phase modulation effects due to the intensity-dependent refractive index of the fiber exactly compensate for the pulse-spreading effects of group-velocity dispersion. Optical solitons have been observed in many physical settings, such as monomode fibers [2], femtosecond lasers [3], and bulk optical materials [4] in which spatial diffraction takes the role of dispersion. What makes optical solitons particularly attractive for applications in high-bit rate all-optical long-distance communication systems is their remarkable robustness. The reason for the soliton robustness is that the wave number of the soliton is distinct from that of the linear dispersive wave, so that there is no energy exchange between them [5].

The propagation of picosecond optical pulses in monomode optical fibers is modeled by the nonlinear Schrödinger (NLS) equation [1]. This equation governs the general situation of dispersive propagation of a pulse envelope with a high carrier frequency in a weakly nonlinear medium. Although the NLS equation includes only two physical effects, group-velocity dispersion and self-phase modulation, it describes a variety of nonlinear optical phenomena. Depending on the relative signs of linear group-velocity dispersion and nonlinearly induced self-phase modulation, they combine to allow bright solitons [6], modulational instability [7], and dark solitons [8]. The NLS equation was generalized to a system of two coupled NLS equations describing the pulse propagation in birefringent optical fibers in the slowly varying envelope approximation [9].

In order to increase the bit rate in fiber optic communication systems to more than 100 Gbit/s for a single-carrier frequency, it is necessary to decrease the pulse width. However, as light pulses become shorter, the standard NLS equation becomes inadequate. Thus additional terms which describe the effects of third-order dispersion and self-steepening must be added to that equation.

The model equation [10–15] for the complex pulse envelope amplitude of the light wave in monomode optical fibers in the subpicosecond-femtosecond domain is the higher-order NLS equation

$$i\frac{\partial q}{\partial Z}+\frac{1}{2}\frac{\partial^2 q}{\partial T^2}+i\varepsilon\frac{\partial^3 q}{\partial T^3} - N^2\left[|q|^2q+i\alpha_1|q|^2\frac{\partial q}{\partial T}+i\alpha_2 q\frac{\partial}{\partial T}(|q|^2)\right]=0, \quad (1)$$

where

$$Z=\frac{|\beta_2|z}{T_0^2},\quad T=\frac{t-z/v_g}{T_0},\quad N^2=\frac{n_2\omega_0P_0T_0^2}{cA_{\text{eff}}|\beta_2|}, \quad (2)$$

$$\varepsilon=\frac{|\beta_3|}{6|\beta_2|T_0},\quad \alpha_1=\frac{2}{\omega_0T_0}+\frac{n'}{nT_0}+\frac{2r'}{rT_0},$$

$$\alpha_2=\frac{2}{\omega_0T_0}+\frac{n'}{nT_0}+\frac{4r'}{rT_0}. \quad (3)$$

In relations (2) and (3), β_2 is the group-velocity dispersion coefficient, β_3 is the third-order dispersion coefficient, v_g is the group velocity, T_0 is the pulse width ($T_{\text{FWHM}}=1.763T_0$), n is the linear index of refraction, n_2 is the Kerr nonlinearity coefficient, ω_0 is the carrier frequency, c is the velocity of light, $A_{\rm eff}$ is the effective core area, P_0 is the peak power of the input pulse, and r is the frequency-dependent radius of the fiber mode. Primes denote the derivative with respect to frequency, and all parameters are evaluated at carrier frequency ω_0. Equation (1) does not take into account either the Raman effect [15] and the fiber loss. The latter is assumed to be overall compensated for by the technique of distributed amplification. In this paper we consider only the case of anomalous second-order dispersion (the group-velocity dispersion coefficient β_2 is negative) and negative third-order dispersion (β_3 is negative). For this choice of fiber parameters we give exact bright solitary-wave solutions of the higher-order NLS equation (1).

Recently, Eq. (1) was generalized to a set of two coupled higher-order NLS equations which can be derived from the Maxwell equations in order to investigate the effects of birefringence on pulse propagation in the femtosecond regime [16–20]. In the present paper we apply the Painlevé singularity structure analysis [21–27] to the fairly general higher-order NLS equation (1) in order to find whether this nonlinear partial differential equation passes the Painlevé integrability test. We mention that the Painlevé singularity structure analysis was previously performed to both the standard NLS equation [28] and the system of two coupled NLS equations describing pulse propagation in birefringent fibers [29]. It was demonstrated in Ref. [28] that the standard NLS equation passes the Painlevé test and the associated Bäcklund transformation and the Hirota bilinearization were constructed. For the system of two coupled NLS equations it was shown that there were only two choices of parameter values for which the system possesses the Painlevé property [29].

In Sec. II, we perform in detail the Painlevé integrability test as it was introduced by Weiss, Tabor, and Carnevale [23], and we arrive at the conclusion that the higher-order NLS equation (1) passes the Painlevé test only for two choices of the ratios of the parameters $\varepsilon:\alpha_1:\alpha_2=1:6:0$ (the Hirota equation [30]), and 1:6:3 (the Sasa-Satsuma equation [31,32]). Then in Sec. III we find a bright single solitary-wave solution of the higher-order NLS equation (1) for arbitrary values of the parameters $\varepsilon,\ \alpha_1$, and α_2 [33,34]. This general solution reduces to those recently reported for the particular cases $\varepsilon:\alpha_1:\alpha_2=1:6:6$ [20], [35,36] and 1:2:2 [35]. We also discuss the appropriateness of these particular solitary waves for describing subpicosecond-femtosecond pulse propagation in monomode optical fibers. We have found that only in the case when the parameters $\varepsilon,\ \alpha_1$, and α_2 are chosen in such a way that $\varepsilon:\alpha_1:\alpha_2=1:(2+\delta_1):(2+\delta_2)$, where δ_1 and δ_2 are either zero or small quantities, the exact single solitary wave properly describes an optical pulse, having either a small or even zero-frequency shift with respect to the carrier frequency, that is, in the frequency region of validity of the extended NLS equation (1).

II. PAINLEVÉ SINGULARITY STRUCTURE ANALYSIS

The Painlevé analysis is one of the systematic methods to identify the integrable cases of the nonlinear partial differential equations [21–27]; that is, to check whether the solutions are free from movable critical manifolds. In order to check for the integrability of partial differential equations we analyze whether these equations have the Painlevé property as it was introduced by Weiss, Tabor, and Carnevale [23]. The method involves expanding the solution in a Laurent series about a singular or pole manifold. Also, the method gives rise to a powerful formalism from which one may deduce the Lax pairs, the Bäcklund transformations, the Hirota equations, the motion invariants, symmetries and commuting flows, and the geometrical structure of the phase space [27].

Next we take the soliton number $N=1$ and perform the Painlevé test of Eq. (1) for the general case $\varepsilon:\alpha_1:\alpha_2=1:\sigma_1:\sigma_2$. Based on physical grounds we choose the parameters σ_1 and σ_2 as positive quantities. We mention that for the next four choices of the parameters $\varepsilon:\alpha_1:\alpha_2=1:6:0$ [30], 1:6:3 [31,32], 1:2:2 [35], and 1:6:6 [20,35,36] exact solutions of the extended NLS equation (1) have been provided without first performing its Painlevé singularity structure analysis. Based on the results of the Painlevé integrability test we show that only the choices $\varepsilon:\alpha_1:\alpha_2=1:6:0$ and 1:6:3 allow the solutions to be true solitons in a strict mathematical sense.

For simplicity we rewrite Eq. (1) as

$$i\frac{\partial q}{\partial z}+\frac{1}{2}\frac{\partial^2 q}{\partial t^2}+|q|^2 q+i\varepsilon\left[\frac{\partial^3 q}{\partial t^3}+\sigma_1|q|^2\frac{\partial q}{\partial t}+\sigma_2 q\frac{\partial}{\partial t}(|q|^2)\right]=0 \tag{4}$$

In order to perform the Painlevé test of Eq. (4), we first put $q=a+ib$, where a and b are, respectively, the real part and the imaginary part of q. Then we obtain the following coupled partial differential equations:

$$a_z+\tfrac{1}{2}b_{tt}+b(a^2+b^2)+\varepsilon\{a_{ttt}+[(\sigma_1+2\sigma_2)a^2+\sigma_1 b^2]a_t+2\sigma_2 abb_t\}=0, \tag{5}$$

$$-b_z+\tfrac{1}{2}a_{tt}+a(a^2+b^2)-\varepsilon\{b_{ttt}+[(\sigma_1+2\sigma_2)b^2+\sigma_1 a^2]b_t+2\sigma_2 aba_t\}=0. \tag{6}$$

The Painlevé analysis in the formulation of Ref. [23] essentially consists of three stages: (i) determination of the leading-order behavior, (ii) identifying the resonances, and (iii) verifying that a sufficient number of arbitrary functions exists without the introduction of movable critical singularity manifolds. To start with, let us introduce the generalized Laurent expansions for the two functions a and b,

$$a=\Phi^{p_1}\sum_{j\geqslant 0}a_j(z,t)\Phi^j,\quad b=\Phi^{p_2}\sum_{j\geqslant 0}b_j(z,t)\Phi^j, \tag{7}$$

in the neighborhood of the singular manifold $\Phi(z,t)=0$ with $\Phi_z\neq 0$ and $\Phi_t\neq 0$. With the very simple choice of the expansion variable Φ (Kruskal ansatz [22]): $\Phi(z,t)=t-\Psi(z)$, where $\Psi(z)$ is an arbitrary analytic function of z, the coefficient functions a_j and b_j become functions of z alone, and this makes the calculations as simple as possible.

Assuming the leading order of the solutions in the forms $a=a_0\Phi^{p_1}$ and $b=b_0\Phi^{p_2}$, and introducing these expressions in Eqs. (5) and (6) (considered in the complex domain), we determine the exponents p_1 and p_2 and the coefficients a_0 and b_0 by balancing the dominant terms. We thus obtain

$$p_1-3=3p_1-1=2p_2+p_1-1, \tag{8}$$

$$p_2-3=3p_2-1=2p_1+p_2-1. \tag{9}$$

By solving this system we obtain the unique solution $p_1=p_2=-1$. The expansion coefficients a_0 and b_0 are determined by

$$a_0^2+b_0^2=-\frac{6}{(\sigma_1+2\sigma_2)}. \tag{10}$$

This result indicates that either a_0 or b_0 can be chosen arbitrarily.

Now we find the resonances, that is, the powers at which the arbitrary functions can enter into the series. Thus we substitute the following expressions into the coupled system (5) and (6):

$$a=a_0\Phi^{-1}+a_j\Phi^{j-1}, \tag{11}$$

$$b=b_0\Phi^{-1}+b_j\Phi^{j-1}. \tag{12}$$

Keeping the leading order terms alone, we obtain a linear system of two algebraic equations in a_j and b_j:

$$\{(j-1)(j-2)(j-3)+j[(\sigma_1+2\sigma_2)a_0^2+\sigma_1b_0^2]-(\sigma_1+2\sigma_2)(3a_0^2+b_0^2)\}a_j+2[j\sigma_2-(\sigma_1+2\sigma_2)]a_0b_0b_j=0, \tag{13}$$

$$2[j\sigma_2-(\sigma_1+2\sigma_2)]a_0b_0a_j+\{(j-1)(j-2)(j-3)+j[(\sigma_1+2\sigma_2)b_0^2+\sigma_1a_0^2]-(\sigma_1+2\sigma_2)(3b_0^2+a_0^2)\}b_j=0. \tag{14}$$

In order to have a nontrivial solution for a_j and b_j, we obtain the following compatibility condition which provides the values of resonances:

$$(j+1)j(j-3)(j-4)\left[j^2-6j+5+\frac{12\sigma_2}{(\sigma_1+2\sigma_2)}\right]=0. \tag{15}$$

The NLS equation (1) can pass the Painlevé test only if the resonances are integers. We find that Eq. (15) admits four integer resonances, namely, $j=-1,0,3$, and 4. The other two remaining resonances become integers only for the following parameter values:

(a) *The case* $\sigma_2=0$. In this case the two remaining resonances are $j=1$ and 5.

(b) *The case* $\sigma_2:\sigma_1=1:1$. Here the two remaining resonances are $j=3$ and 3.

(c) *The case* $\sigma_2:\sigma_1=1:2$. In this case the two remaining resonances are $j=2$ and 4.

In order that the Painlevé property is satisfied we have to ensure that sufficient number of arbitrary functions exist at the appropriate integer resonance values. We now briefly discuss the search for arbitrary functions at the resonance values above. The resonance at $j=-1$ corresponds to the arbitrariness of Φ itself.

A. The case $\sigma_2=0$

By imposing that the coefficient of Φ^{-4} vanish we obtain $a_0^2+b_0^2=-6/\sigma_1$. Thus from the leading order terms we have obtained that the resonance at $j=0$ corresponds to the fact that one of the two functions a_0 and b_0 is arbitrary. Next we substitute the Laurent series (7) in the system (5) and (6), and collect the coefficients of different powers of Φ, so that we can evaluate the further coefficients. By collecting the coefficients of Φ^{-3} in Eqs. (5) and (6), we obtain

$$a_0^2a_1+a_0b_0b_1=\frac{b_0\left(1-\dfrac{6}{\sigma_1}\right)}{2\varepsilon\sigma_1}, \tag{16}$$

$$a_0b_0a_1+b_0^2b_1=-\frac{a_0\left(1-\dfrac{6}{\sigma_1}\right)}{2\varepsilon\sigma_1}. \tag{17}$$

Because $j=1$ is a resonance, one of the functions a_1 and b_1 should be arbitrary in order that Eq. (4) possesses the Painlevé property. Thus imposing that both Eqs. (16) and (17) coincide up to a factor, we obtain the following compatibility condition:

$$(a_0^2+b_0^2)\left(1-\frac{6}{\sigma_1}\right)=0\,. \tag{18}$$

From Eq. (18) we obtain $\sigma_1=6$. Thus we arrive at the well-known Hirota equation [30]. We shall verify that in this case the Laurent expansion admits six arbitrary functions, and therefore the Hirota equation passes the Painlevé test as expected.

Proceeding further in this way and collecting the coefficients of Φ^{-2} in Eqs. (5) and (6) one obtains

$$a_2=\frac{b_1}{6\varepsilon}-\frac{b_1^2}{a_0}+\frac{a_0\Phi_z}{6\varepsilon}, \tag{19}$$

$$b_2=\frac{b_1b_0}{6\varepsilon a_0}-\frac{b_0b_1^2}{a_0^2}+\frac{b_0\Phi_z}{6\varepsilon}; \tag{20}$$

thus both a_2 and b_2 are fixed. This result is consistent with the lack of the resonance $j=2$.

Next, by collecting the coefficients of Φ^{-1} in Eqs. (5) and (6), the following relation is obtained:

$$a_3=\frac{a_0}{b_0}b_3+\frac{b_1}{24\varepsilon^2 a_0 b_0}-\frac{b_1^2}{6\varepsilon a_0^2 b_0}-\frac{a_{0z}}{12\varepsilon b_0^2}+\frac{\Phi_z}{24\varepsilon^2 b_0}. \tag{21}$$

Thus one of the quantities a_3 or b_3 is arbitrary ($j=3$ is a resonance). By collecting the coefficients of Φ^0 in Eqs. (5) and (6), we obtain

$$a_4=\frac{a_0}{b_0}b_4-\frac{b_1}{144\varepsilon^3 b_0^2}+\frac{b_1^2}{24\varepsilon^2 a_0 b_0^2}+\frac{b_3}{6\varepsilon b_0^2}+\frac{a_0 a_{0z}}{72\varepsilon^2 b_0^3}-\frac{b_1 a_{0z}}{12\varepsilon a_0^2 b_0}+\frac{b_{1z}}{12\varepsilon a_0 b_0}-\frac{a_0\Phi_z}{144\varepsilon^3 b_0^2}. \tag{22}$$

This relation indicates that either a_4 or b_4 is an arbitrary function, as it should be, because $j=4$ is a resonance.

Finally, by collecting the coefficients of Φ^1 in Eqs. (5) and (6), and by using all the above relationships, one can easily see that one of the functions a_5 and b_5 can be chosen arbitrarily, a result consistent with the fact that $j=5$ is a resonance. In this way we established that there exists the required number of arbitrary functions corresponding to the resonance values $j=-1$, 0, 1, 3, 4, and 5. In conclusion, for the choice of coefficients $\varepsilon:\alpha_1:\alpha_2=1:6:0$, Eq. (1) passes the Painlevé test.

B. The case $\sigma_1=\sigma_2$

In this case we obtain the relationship $a_0^2+b_0^2=-2/\sigma_2$ between the coefficients of the leading order terms of the Laurent expansion; thus one of the functions a_0, b_0 is arbitrary. Here the resonances are the following integer numbers: $j=-1$, 0, 3, 3, 3, and 4. Now we briefly discuss the search for arbitrary functions at the resonance values above. Collecting the coefficients of Φ^{-3} in the system (5) and (6), we obtain a linear system with the unique solution

$$a_1=-\frac{b_0(\sigma_2-2)}{4\sigma_2\varepsilon}, \tag{23}$$

$$b_1=\frac{a_0(\sigma_2-2)}{4\sigma_2\varepsilon}. \tag{24}$$

This is consistent with the lack of the resonance $j=1$. By proceeding further in this way and collecting the coefficients of Φ^{-2}, we obtain the values of a_2 and b_2:

$$a_2=-\frac{(\sigma_2^2-8\sigma_2+12)}{48\sigma_2^2\varepsilon^2}a_0+\frac{a_0\Phi_z}{6\varepsilon}, \tag{25}$$

$$b_2=-\frac{(\sigma_2^2-8\sigma_2+12)}{48\sigma_2^2\varepsilon^2}b_0+\frac{b_0\Phi_z}{6\varepsilon}. \tag{26}$$

This result is consistent with the fact that $j=2$ is not a resonance. Finally collecting the coefficients of Φ^{-1} the following system is obtained:

$$a_{0z}-\frac{b_0(\sigma_2-2)}{2\sigma_2^3\varepsilon^2}+\frac{b_0\Phi_z}{\sigma_2\varepsilon}=0, \tag{27}$$

$$b_{0z}+\frac{a_0(\sigma_2-2)}{2\sigma_2^3\varepsilon^2}+\frac{a_0\Phi_z}{\sigma_2\varepsilon}=0. \tag{28}$$

By solving this system we are left with a_0 and b_0, both of them being fixed. This contradiction shows that the perturbed NLS equation (1) with the coefficients $\varepsilon:\alpha_1:\alpha_2=1:\sigma_2:\sigma_2$ does not pass the Painlevé test. This result is intimately related to the fact that in this case the degree of multiplicity of the resonance $j=3$ exceeds the dimension of the appropriate vectorial space (equal to 2 in this case). As a particular result we obtain that Eq. (1) with the coefficients $\varepsilon:\alpha_1:\alpha_2=1:2:2$ and 1:6:6 [35,36] does not pass the Painlevé test for integrability.

C. The case $\sigma_1=2\sigma_2$

For this choice of the parameters we find $a_0^2+b_0^2=-3/2\sigma_2$, thus one of these functions being arbitrary. In this case the resonances are the following integers: $j=-1$, 0, 2, 3, 4, and 4. Collecting the coefficients of Φ^{-3} in Eqs. (5) and (6), we are left with a linear system with the unique solution

$$a_1=-\frac{b_0(2\sigma_2-3)}{6\sigma_2\varepsilon}, \tag{29}$$

$$b_1=\frac{a_0(2\sigma_2-3)}{6\sigma_2\varepsilon}. \tag{30}$$

Next by collecting the coefficients of Φ^{-2}, we obtain

$$a_2=-\frac{b_0}{a_0}b_2+\frac{(2\sigma_2-3)(\sigma_2-3)}{24\sigma_2^3\varepsilon^2 a_0}-\frac{\Phi_z}{4\sigma_2\varepsilon a_0}. \tag{31}$$

This result implies that one of the functions a_2 and b_2 is arbitrary ($j=2$ is a resonance). Collecting the coefficients of Φ^{-1}, we obtain a linear system of equations for a_3 and b_3, and, imposing that these equations should coincide up to a factor, because $j=3$ is a resonance, we are left with the following compatibility condition:

$$(\sigma_2-3)(a_0^2+b_0^2)=0. \tag{32}$$

Thus we obtain $\sigma_2=3$ and the relationship between a_3 and b_3:

$$a_3=\frac{a_0}{b_0}b_3+\frac{1}{432\varepsilon^3 b_0}-\frac{a_{0z}}{6\varepsilon b_0^2}+\frac{\Phi_z}{24\varepsilon^2 b_0}. \tag{33}$$

By collecting the coefficients of Φ^0 in Eqs. (5) and (6), and introducing the expressions of a_j and b_j ($j=0$, 1, 2, and 3), we find that both a_4 and b_4 can be chosen arbitrarily, a result consistent with the fact that the degree of multiplicity of the resonance $j=4$ is 2. Thus the perturbed NLS equation (1) with the coefficients $\varepsilon:\alpha_1:\alpha_2=1:6:3$ passes the Painlevé test. We notice that both the single and multiple soliton solutions of this equation were provided by the inverse scattering method [31,32].

III. BRIGHT SOLITARY WAVE SOLUTIONS OF THE HIGHER-ORDER NLS EQUATION

For soliton number $N=1$ and for arbitrary values of the parameters ϵ, α_1, and α_2, a bright solitary wave solution [33,34] of Eq. (1) can be written in the following form:

$$q(T,Z)=A\ \mathrm{sech}[\eta(T-v^{-1}Z-T_0)] \times\exp(-i\xi T+i\kappa Z+i\varphi_0), \tag{34}$$

where

$$A=\eta\left(\frac{6\varepsilon}{\alpha_1+2\alpha_2}\right)^{1/2}, \quad v^{-1}=\varepsilon(\eta^2-3\xi^2)-\xi,$$

$$\kappa=\varepsilon\xi(3\eta^2-\xi^2)+\frac{1}{2}(\eta^2-\xi^2),$$

and

$$\xi=\frac{6\varepsilon-\alpha_1-2\alpha_2}{12\varepsilon\alpha_2}.$$

We supposed here that in the anomalous dispersion region both second- and third-order dispersion coefficients are negative (ε is a positive number). We notice that unlike the standard NLS equation the quantity ξ, which is proportional to the frequency shift $\Delta\omega_0$, is a fixed parameter. The solitary wave (34) is not a soliton in a strict mathematical sense for every ratios of the parameters ε, α_1, and α_2 but for $\varepsilon:\alpha_1:\alpha_2=1:6:0$ (the Hirota equation [30]), and 1:6:3 (the Sasa-Satsuma equation [31,32]).

Recently, by using the Hirota method [30] both bright and dark solutions of the higher-order NLS equation (1) were found for the case when $\varepsilon:\alpha_1:\alpha_2=1:6:6$ [35,36] (see also Ref. [20] for a generalization to two coupled NLS equations with higher-order terms). Due to the fact that in this case the higher-order NLS equation does not pass the Painlevé test, such solutions are not solitons in the strict mathematical sense, and should be referred to more accurately as solitary waves.

We see from relations (3) that the coefficients of higher-order nonlinear terms are approximately equal $\alpha_1\simeq\alpha_2$. In the following we consider that the parameters ε, α_1, and α_2 are such that $\varepsilon:\alpha_1:\alpha_2=1:(2+\delta_1):(2+\delta_2)$. In this case the amplitude A and the parameter ξ of the solitary wave (34) become $[1-(\delta_1+2\delta_2)/(6+\delta_1+2\delta_2)]^{1/2}\eta$ and $-[(\delta_1+2\delta_2)/12\varepsilon(2+\delta_2)]$, respectively. The solitary wave (34) exhibits a frequency shift $\Delta\omega_0=[(\delta_1+2\delta_2)/24]\omega_0$. In the particular case $\varepsilon:\alpha_1:\alpha_2=1:6:6$, that is, $\delta_1=\delta_2=4$, the corresponding carrier frequency shift is $\Delta\omega_0=\omega_0/2$. This large frequency shift renders the exact bright solitary wave solution of the 1:6:6 model, obtained by use of the Hirota method, not suitable for describing bright subpicosecond-femtosecond soliton propagation in monomode optical fibers. Thus the exact solution given in [35] for the 1:6:6 model describes a pulse with a frequency shift which takes it well out of the region of validity of Eq. (1).

When $|\delta_1|,|\delta_2|\ll 1$, the frequency shift $\Delta\omega_0=[(\delta_1+2\delta_2)/24]\omega_0$ is much less than the carrier frequency ω_0. Therefore, in this case, the exact solution (34) describes a solitary wave in the region of validity of the extended NLS equation (1), that is, in the frequency region in the neighborhood of the chosen carrier frequency. It should be mentioned that these real pulse-type solutions of the NLS equation (1) can never be true solitons because the regimes of integrability and of validity of the pulse-type solutions are mutually exclusive.

In the particular case $\delta_1=\delta_2=0$, that is, for the ratios 1:2:2 of the coefficients ε, α_1, and α_2, the solitary wave (34) does not exhibits any frequency shift. It becomes the following bright solitary wave solution of the corresponding extended NLS equation which was also found in [35] by using the Hirota direct method:

$$q(T,Z)=\eta\ \mathrm{sech}[\eta(T-v^{-1}Z-T_0)]\exp(i\kappa Z+i\varphi_0), \tag{35}$$

where $v^{-1}=\epsilon\eta^2$, and $\kappa=\frac{1}{2}\eta^2$. This particular solution properly describe the propagation of ultra short pulses in monomode optical fibers having no shift in frequency with respect to the carrier frequency.

It is of interest to compare the bright single soliton of the standard NLS equation

$$q(T,Z)=\eta\ \mathrm{sech}[\eta(T+\xi Z-T_0)] \times\exp[-i\xi T+i\tfrac{1}{2}(\eta^2-\xi^2)Z+i\varphi_0] \tag{36}$$

to the bright solitary wave (34) of the higher-order NLS equation (1). It should be noticed that in Eq. (36) ξ and η are arbitrary parameters. However, the true soliton (36) describes a pulse with a frequency shift proportional to the soliton parameter ξ. Thus the parameter ξ has to be small in order to assure a small frequency shift with respect to the carrier frequency. The two-parameter family of soliton solutions (36) has the salient feature that the normalized peak intensity I (the square of the soliton amplitude) and the soliton width $\tau=1/\eta$ obey the relationship $I\tau^2=1$. For bright solitary wave solutions (34) of the higher-order NLS equation (1) the product $I\tau^2=[1+(\delta_1+2\delta_2)/6]^{-1}<1$. Thus, the product $I\tau^2$ attains its maximum value, equal to 1, for $\delta_1=\delta_2=0$; that is, for the particular solitary wave (35) that corresponds to the ratios 1:2:2 of the coefficients ε, α_1, and α_2. These ratios can be achieved, for example, for the following two sets of fiber parameters: $\beta_2=-2.5$ ps^2/km, $\beta_3=-0.012$ ps^3/km or $\beta_2=-0.5$ ps^2/km, $\beta_3=-0.0024$ ps^3/km at $\lambda=1.55$ μm. Although the third-order dispersion coefficient β_3 has to be negative and very small, the drawing of such dispersion-shifted fibers is technically possible [37], and we expect that the simplest form (35) of the bright solitary wave (34) could be observed experimentally in a properly tailored optical fiber.

IV. CONCLUSIONS

We performed the Painlevé singularity structure analysis to the general higher-order NLS equation (1) in order to conclude whether this nonlinear partial differential equation passes the Painlevé integrability test. The main result of this paper is that the NLS equation (1) fails to pass the integrability test for almost the entire parameter space. Moreover, we have shown that two recently introduced higher-order

NLS equations [35,36] fail to pass the Painlevé test. We have also shown that some of the bright solitary waves given in Refs. [20] and [35] by using the Hirota direct method, describe pulses with large frequency shifts. Thus those exact solutions are inappropriate for describing subpicosecond-femtosecond soliton propagation in monomode optical fibers. Another higher-order nonlinear Schrödinger equation is introduced, and bright solitary waves are given. These solitary waves properly describe optical pulses with very small frequency shifts. There are also given typical values of fiber parameters for the existence of these subpicosecond-femtosecond bright solitary waves.

Finally, we mention that the results contained in this paper can be rather easily extended to dark pulses in monomode optical fibers. In this case we should consider that β_2 and β_3 are positive quantities (positive second-order dispersion and positive third-order dispersion). The corresponding extended NLS equation differs from Eq. (1) only by the signs of the coefficients in front of the second and third terms in Eq. (1). We can anticipate that this new extended NLS equation passes the Painlevé test only if the ratio of its coefficients is either $-\varepsilon:\alpha_1:\alpha_2=-1:6:3$ or $-1:6:0$. Thus dark soliton solutions of these higher-order NLS equations could be obtained by using the powerful inverse scattering transform formalism and these results will be published elsewhere.

Note added. The article by M. Gedalin, T. C. Scott, and Y. B. Band [Phys. Rev. Lett. **78**, 448 (1997)] was recently brought to the authors' attention. This article performed the same singularity analysis as the current paper, arriving at the same results, as well as using the Hirota method to derive analytical solutions.

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